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Trace operators for modulation, α-modulation and Besov spaces

Hans G. Feichtinger [a], Chunyan Huang [b], Baoxiang Wang [c,*]

[a] Faculty of Mathematics, University of Vienna, Vienna A-1090, Austria
[b] Inst. G.P.M., RWTH Aachen, Templergraben 55, D-52056 Aachen, Germany
[c] LMAM, School of Mathematical Sciences, Peking University, Beijing 100871, China

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ABSTRACT

In this paper, we consider the trace theorem for modulation spaces $M^s_{p,q}$, α-modulation spaces $M^{s,\alpha}_{p,q}$ and Besov spaces $B^s_{p,q}$. For the modulation space, we obtain the sharp results.

1. Introduction

There are two kinds of basic coverings on Euclidean $\mathbb{R}^n$ which is very useful in the theory of function spaces and their applications, one is the uniform covering $\mathbb{R}^n = \bigcup_{k\in\mathbb{Z}^n} Q_k$, where Q_k denotes the unit cube with center k; another is the dyadic covering $\mathbb{R}^n = \bigcup_{k\in\mathbb{N}}\{\xi\colon\ 2^{k-1} \leqslant |\xi| < 2^k\} \cup \{\xi\colon\ |\xi| \leqslant 1\}$. Roughly speaking, these decompositions together with the frequency-localized techniques yield the frequency-uniform decomposition operator $\Box_k \sim \mathscr{F}^{-1}\chi_{Q_k}\mathscr{F}$ and the dyadic decomposition operator $\Delta_k \sim \mathscr{F}^{-1}\chi_{\{\xi\colon\ |\xi|\sim 2^k\}}\mathscr{F}$, respectively. The tempered distributions acted on these decomposition operators and having finite $\ell^q(L^p(\mathbb{R}^n))$ norms form Feichtinger's modulation spaces and Besov spaces, respectively.

During the past twenty years, the third covering was independently found by Feichtinger and Gröbner [3,4,10], and Päivärinta and Somersalo [15]. This covering, so-called α-covering has a moderate scale which is rougher than that of the uniform covering and is thinner than that of the dyadic covering.

Applying the α-covering to the frequency spaces, in a similar way as the definition of Besov spaces, Gröbner [10] introduced the notion of α-modulation spaces, which contain Besov spaces $B^s_{p,q}$ and modulation spaces $M^s_{p,q}$ as special spaces in the limit cases $\alpha = 1$ and $\alpha = 0$, respectively.

Let $n \geqslant 2$. For any $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we denote $\bar{x} = (x_1, \dots, x_{n-1})$. Given a Banach function space $X(\mathbb{R}^n)$ defined on $\mathbb{R}^n$ and $f \in X$, we ask for the trace of f on the hyperplane $\{x\colon\ x = (\bar{x}, 0)\}$. For the sake of convenience, this hyperplane will be written as $\mathbb{R}^{n-1}$. We note that for a tempered distribution f defined on $\mathbb{R}^n$, $f(\bar{x}, 0)$ has no straightforward meaning and the question is how to define the trace for a class of tempered distributions. One can resort to the Schwartz function φ, which has a pointwise trace $\varphi(\bar{x}, 0)$. It can be extended to (quasi-)Banach function spaces which contain the Schwartz space $\mathscr{S}$ as a dense subspace. It is clear that a clarification of this problem is of importance for the boundary value problems of the partial differential equations; cf. [13,14,19]. Now we exactly describe the trace operators.

* Corresponding author.
E-mail addresses: hans.feichtinger@univie.ac.at (H.G. Feichtinger), chunyanh80@gmail.com (C. Huang), wbx@math.pku.edu.cn (B. Wang).

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Definition 1.1. Let X and Y be quasi-Banach function spaces defined on $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$, respectively. Assume that the Schwartz space $\mathscr{S}$ is dense in X. Denote

$$\mathbb{T}: f(x) \to f(\bar{x}, 0), \quad f \in \mathscr{S}. \tag{1.1}$$

Assuming that there exists a constant $C > 0$ such that

$$\|\mathbb{T}f\|_Y \leqslant C\|f\|_X, \quad \forall f \in \mathscr{S}, \tag{1.2}$$

one can extend $\mathbb{T}: X \to Y$ by the density of $\mathscr{S}$ in X and we write $f(\bar{x}, 0) = \mathbb{T}f$, which is said to be the trace of $f \in X$. Moreover, if there exists a continuous linear operator $\mathbb{T}^{-1}: Y \to X$ such that $\mathbb{T}\mathbb{T}^{-1}$ is the identity operator on Y, then $\mathbb{T}$ is said to be a trace-retraction from X onto Y.

If $\mathscr{S}$ is not dense in X, we cannot define the trace operator in a general way, see [17] for instance. If $\mathbb{T}$ is a trace-reaction from X onto Y, we see that the trace of $f \in X$ is well behaved in Y. The trace theorems in modulation spaces and Besov spaces have been extensively studied. Feichtinger [5] considered the trace theorem for the modulation space $M^s_{p,q}$ in the case $1 \leqslant p, q \leqslant \infty$, $s > 1/q'$ and he obtained that $\mathbb{T}M^s_{p,q}(\mathbb{R}^n) = M^{s-1/q'}_{p,q}(\mathbb{R}^{n-1})$. Frazier and Jawerth [8] proved that $\mathbb{T}B^s_{p,q}(\mathbb{R}^n) = B^{s-1/p}_{p,q}(\mathbb{R}^{n-1})$ in the case $0 < p, q \leqslant \infty$ and $s - 1/p > \max((n-1)(1/p - 1), 0)$. Recently, Schneider [17] has improved the results of the trace of Besov spaces $\mathfrak{B}^s_{p,q}$-characterized via atomic decompositions and obtained that $\mathbb{T}\mathfrak{B}^s_{p,q}(\mathbb{R}^n) = \mathfrak{B}^{s-1/p}_{p,q}(\mathbb{R}^{n-1})$ for $s > 1/p$, $\mathbb{T}\mathfrak{B}^{1/p}_{p,q}(\mathbb{R}^n) = L^p(\mathbb{R}^{n-1})$ for $q \leqslant 1 \wedge p$. Similar results also hold for Triebel–Lizorkin spaces and for Besov spaces defined in a C^k domain Ω, cf. [17,18].

Our aim is to study the trace operators for modulation, α-modulation and Besov spaces. Since we are unable to get a general way to define the trace in the cases $p, q = \infty$, we will always assume that $0 < p, q < \infty$, though our results also hold for $p, q = \infty$ if s is suitably large. For the modulation space, we get that $\mathbb{T}M^s_{p,q,p\wedge q\wedge 1}(\mathbb{R}^n) = M^s_{p,q}(\mathbb{R}^{n-1})$ for all $s \in \mathbb{R}$, $0 < p, q < \infty$, where $M^s_{p,q,p\wedge q\wedge 1}$ is a kind of anisotropic modulation spaces, see Section 2 for some further comments. The new characteristic for the trace of the modulation spaces differ from Feichtinger's classical result and the cases $s \leqslant 0$ and $p, q < 1$ are also involved in our new results.

For the α-modulation spaces, we will show that $\mathbb{T}: M^{s+\alpha/p,\alpha}_{p,p\wedge q\wedge 1}(\mathbb{R}^n) \to M^{s,\alpha}_{p,q}(\mathbb{R}^{n-1})$ with some additional conditions on p, q and α. Taking $\alpha = 0$, we have $\mathbb{T}: M^s_{p,p\wedge q\wedge 1}(\mathbb{R}^n) \to M^s_{p,q}(\mathbb{R}^{n-1})$, which is a strictly weaker version of the result on modulation spaces in the case $p, q > 1$. On the other hand, considering another limit case $\alpha = 1$, we have $\mathbb{T}: B^{s+1/p}_{p,p\wedge q\wedge 1}(\mathbb{R}^n) \to B^s_{p,q}(\mathbb{R}^{n-1})$, which is also slightly weaker than that of the result in Frazier and Jawerth [8] and Schneider [17]. So, our result on α-modulation spaces seems not optimal.

For Besov spaces, we will generalize the results of Frazier and Jawerth [8] and Schneider [17] to the case $s - 1/p < 0$.

This paper is organized as follows. In Section 2, we present a new α-covering and an equivalent norm on α-modulation spaces. Moreover, in order to characterize the trace of modulation spaces, the anisotropic modulation spaces are introduced. The main results will be stated in Section 3. Finally, we prove our main results in Sections 4–6.

The following are some notations which will be frequently used in this paper: $c < 1$, $C > 1$ will denote positive universal constants, which can be different at different places. $a \lesssim b$ stands for $a \leqslant Cb$ for some constant $C > 1$, $a \sim b$ means that $a \lesssim b$ and $b \lesssim a$. We write $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$. We denote by p' the dual number of $p \in [1, \infty]$, i.e., $1/p + 1/p' = 1$. We will use Lebesgue spaces $L^p := L^p(\mathbb{R}^n)$, $\|\cdot\|_p := \|\cdot\|_{L^p}$. We denote by $\mathscr{S} := \mathscr{S}(\mathbb{R}^n)$ and $\mathscr{S}' := \mathscr{S}'(\mathbb{R}^n)$ the Schwartz space and tempered distribution space, respectively. $B(x, R)$ stands for the ball in $\mathbb{R}^n$ with center x and radius R, $Q(x, R)$ denote the cube in $\mathbb{R}^n$ with center x and side-length $2R$. $\mathscr{F}$ or $\widehat{\ }$ denotes the Fourier transform; $\mathscr{F}^{-1}$ denotes the inverse Fourier transform. For any set A with finite elements, we denote by $\#A$ the number of the elements of A.

2. α-modulation spaces

2.1. Definition

A countable set $\mathcal{Q}$ of subsets $Q \subset \mathbb{R}^n$ is said to be an admissible covering if $\mathbb{R}^n = \bigcup_{Q\in\mathcal{Q}} Q$ and there exists $n_0 < \infty$ such that $\#\{Q' \in \mathcal{Q}\colon Q \cap Q' \neq \varnothing\} \leqslant n_0$ for all $Q \in \mathcal{Q}$. Denote

$$\begin{aligned} r_Q &= \sup\bigl\{r \in \mathbb{R}\colon B(c_r, r) \subset Q\bigr\}, \\ R_Q &= \inf\bigl\{R \in \mathbb{R}\colon Q \subset B(c_R, R)\bigr\}. \end{aligned} \tag{2.1}$$

Let $0 \leqslant \alpha \leqslant 1$. An admissible covering is called an α-covering of $\mathbb{R}^n$, if $|Q| \sim \langle x\rangle^{\alpha n}$ (uniformly) holds for all $Q \in \mathcal{Q}$ and for all $x \in Q$, and $\sup_{Q\in\mathcal{Q}} R_Q/r_Q \leqslant K$ for some $K < \infty$.

Let $\mathcal{Q}$ be an α-covering of $\mathbb{R}^n$. A corresponding bounded admissible partition of unity of order p (p-*BAPU*) $\{\psi_Q\}_{Q\in\mathcal{Q}}$ is a family of smooth functions satisfying

$$\begin{cases} \psi_Q : \mathbb{R}^n \to [0,1], \quad \operatorname{supp}\psi_Q \subset Q, \\ \sum_{Q\in\mathcal{Q}} \psi_Q(\xi) \equiv 1 \quad \forall \xi \in \mathbb{R}^n, \\ \sup_{Q\in\mathcal{Q}} |Q|^{1/(p\wedge 1)-1} \|\mathscr{F}^{-1}\psi_Q\|_{L^{(p\wedge 1)}} < \infty. \end{cases} \tag{pBAPU}$$

Definition 2.1. Let $0 < p, q \leqslant \infty$, $s \in \mathbb{R}$, $0 \leqslant \alpha \leqslant 1$. Let $\mathcal{Q}$ be an α-covering of $\mathbb{R}^n$ with the p-BAPU $\{\psi_Q\}_{Q\in\mathcal{Q}}$. We denote by $M^{s,\alpha}_{p,q}$ the space of all tempered distributions f for which the following is finite:

$$\|f\|_{M^{s,\alpha}_{p,q}} = \left(\sum_{Q\in\mathcal{Q}} \langle\xi_Q\rangle^{qs} \|\mathscr{F}^{-1}\psi_Q \mathscr{F} f\|^q_{L^p(\mathbb{R}^n)}\right)^{1/q},$$

where $\xi_Q \in Q$ is arbitrary. For $q = \infty$, we have a usual substitution for the ℓ^q norm with the ℓ^∞ norm (see [1,7]).

We now state an exact equivalent norm on $M^{s,\alpha}_{p,q}$. Denote

$$Q_k = Q\left(|k|^{\frac{\alpha}{1-\alpha}} k, r\langle k\rangle^{\frac{\alpha}{1-\alpha}}\right), \quad k \in \mathbb{Z}^n.$$

It is known that, there exists a constant $r_1 > 0$ such that for any $r > r_1$, $\{Q_k\}_{k\in\mathbb{Z}^n}$ is an α-covering of $\mathbb{R}^n$, i.e., $\mathbb{R}^n = \bigcup_{k\in\mathbb{Z}^n} Q_k$ and there exists $n_0 \in \mathbb{N}$ such that $\#\{l \in \mathbb{Z}^n : Q_k \cap Q_{k+l} \neq \emptyset\} \leqslant n_0$ (cf. [1]). Moreover, $|Q_k| \sim \langle k\rangle^{\frac{n\alpha}{1-\alpha}}$. Let $\eta : \mathbb{R} \to [0,1]$ be a smooth bump function satisfying

$$\eta(\xi) := \begin{cases} 1, & |\xi| \leqslant 1, \\ \text{smooth}, & 1 < |\xi| \leqslant 2, \\ 0, & |\xi| \geqslant 2. \end{cases} \tag{2.2}$$

We write for $k = (k_1, \dots, k_n)$ and $\xi = (\xi_1, \dots, \xi_n)$,

$$\phi_{k_i}(\xi_i) = \eta\left(\frac{\xi_i - |k|^{\frac{\alpha}{1-\alpha}} k_i}{r\langle k\rangle^{\frac{\alpha}{1-\alpha}}}\right).$$

Put

$$\psi_k(\xi) = \frac{\phi_{k_1}(\xi_1)\dots\phi_{k_n}(\xi_n)}{\sum_{k\in\mathbb{Z}^n} \phi_{k_1}(\xi_1)\dots\phi_{k_n}(\xi_n)}, \quad k \in \mathbb{Z}^n. \tag{2.3}$$

We have

Lemma 2.2. *Let $0 \leqslant \alpha < 1$, $0 < p \leqslant \infty$ and $\{\psi_k\}_{k\in\mathbb{Z}^n}$ be as in (2.3). Then $\{\psi_k\}_{k\in\mathbb{Z}^n}$ is a p-BAPU for $r > r_1$. In the case $\alpha = 0$, we can take $r_1 = 1/2$.*

Proposition 2.3. *Let $0 \leqslant \alpha < 1$, $0 < p, q \leqslant \infty$, then*

$$\|f\|_{M^{s,\alpha}_{p,q}} = \left(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{\frac{qs}{1-\alpha}} \|\mathscr{F}^{-1}\psi_k \mathscr{F} f\|^q_{L^p(\mathbb{R}^n)}\right)^{1/q}$$

is an equivalent norm on α-modulation space with the usual modification for $q = \infty$.

Proof. See [1]. □

2.2. Equivalent norm via a new p-BAPU

We now construct a new covering, which is of importance for the proof of Theorem 3.3. We denote by $\{a\}$ the minimal integer which is larger than or equals to a. Let $r_1 > 0$ be a constant which will be chosen in Proposition 2.4 below. Let $j \in \mathbb{Z}\backslash\{0\}$. We divide $[-|j|^{\frac{1}{1-\alpha}}, |j|^{\frac{1}{1-\alpha}}]$ into $2\{|j|/r_1\}$ intervals with equal length. We may assume that $2|j|/r_1 = 2N_j$ and N_j is an integer. Namely,

$$\left[-|j|^{\frac{1}{1-\alpha}}, |j|^{\frac{1}{1-\alpha}}\right] = [r_{j,-N_j}, r_{j,-N_j+1}] \cup \dots \cup [r_{j,N_j-1}, r_{j,N_j}].$$

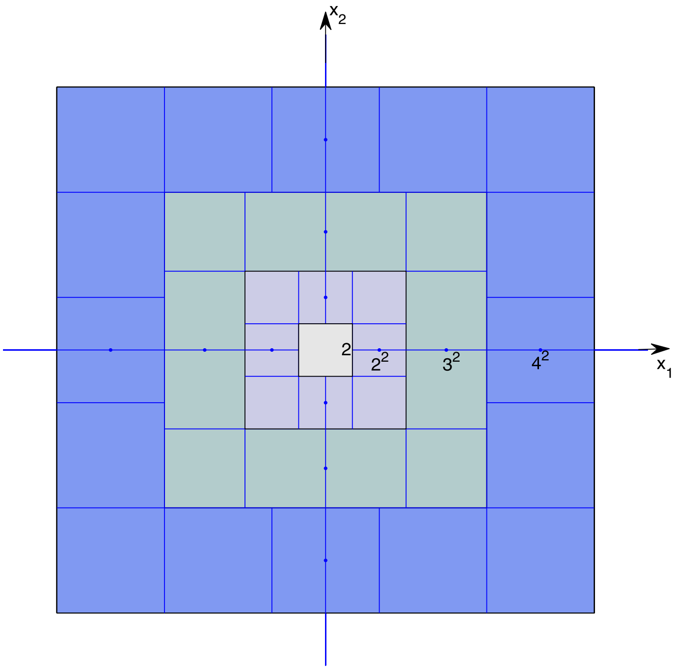

Fig. 1. α-covering, the case $n = 2$, $\alpha = 1/2$, $r_1 = 1$.

Denote

$$\mathscr{R} = \{r_{j,s}\colon\ j \in \mathbb{N},\ s = -N_j, \dots, N_j\}.$$

We further write

$$\mathscr{K}_j^n = \left\{k = (k_1, \dots, k_n)\colon\ k_i \in \mathscr{R},\ \max_{1 \leqslant i \leqslant n} |k_i| = |j|^{\frac{1}{1-\alpha}}\right\}.$$

For any $k \in \mathscr{K}_j^n$, we write

$$Q_{kj} = Q\left(k, r|j|^{\frac{\alpha}{1-\alpha}}\right), \quad Q_{k0} = Q(0, 2).$$

We will write $\mathscr{K}_j = \mathscr{K}_j^n$ if there is no confusion (see Fig. 1).

Proposition 2.4. *There exists $r_1 > 0$ such that for any $r > r_1$, $\{Q_{kj}\}_{k \in \mathscr{K}_j, j \in \mathbb{Z}_+}$ is an α-covering of $\mathbb{R}^n$.*

Proof. Let $j \in \mathbb{N} \cup \{0\}$. We see that there exists $r_1 > 0$ such that for any $r > r_1$, $\{Q(|j|^{\frac{\alpha}{1-\alpha}} j, r|j|^{\frac{\alpha}{1-\alpha}})\}_j$ is an α-covering of $\mathbb{R}$. Hence we easily see that

$$\mathbb{R} \subset \bigcup_{k \in \mathscr{K}_j,\, j \in \mathbb{Z}_+} Q_{kj}, \quad |Q_{kj}| \sim |j|^{\frac{n\alpha}{1-\alpha}} \sim \langle \xi_{Q_{kj}} \rangle^{n\alpha},\ \forall \xi_{Q_{kj}} \in Q_{k_j},$$
$$\#\{Q_{k'j'}\colon\ Q_{kj} \cap Q_{k'j'} \neq \emptyset\} \leqslant n_0 < \infty. \qquad \square$$

Now, on the basis of the α-covering constructed above, we further construct a p-BAPU. Let j be fixed. Denote for $i = 1, \dots, n$,

$$\phi_{kj}(\xi_i) = \phi\left(\frac{\xi_i - k_i}{r\langle j\rangle^{\frac{\alpha}{1-\alpha}}}\right), \quad k = (k_1, \dots, k_n) \in \mathscr{K}_j,$$
$$\phi_{kj}(\xi) = \phi_{kj}(\xi_1) \dots \phi_{kj}(\xi_n).$$

We put

$$\psi_{kj}(\xi) = \frac{\phi_{kj}(\xi)}{\sum_{k \in \mathscr{K}_j,\, j \in \mathbb{Z}_+} \phi_{kj}(\xi)}. \tag{2.4}$$

Proposition 2.5. *Let $0 < p < \infty$, ψ_{kj} be as in (2.4). Then $\{\psi_{kj}\}_{k\in\mathscr{K}_j,\, j\in\mathbb{Z}_+}$ is a p-BAPU.*

Proof. It suffices to show the third condition in (*pBAPU*), which can be shown by following the same way as in [1]. □

Noticing that $|\xi| \sim |j|^{1/(1-\alpha)}$ if $\xi \in \mathscr{K}_j,\ j \neq 0$, we immediately have

Proposition 2.6. *Let $0 < \alpha < 1$, $0 < p, q \leqslant \infty$, then*

$$\|f\|_{M^{s,\alpha}_{p,q}} = \left(\sum_{j\in\mathbb{Z}_+} \langle j\rangle^{sq/(1-\alpha)} \sum_{k\in\mathscr{K}_j} \left\|\mathscr{F}^{-1}\psi_{kj}\mathscr{F}f\right\|^q_{L^p(\mathbb{R}^n)}\right)^{1/q}$$

is another equivalent norm on α-modulation space.

2.3. Modulation spaces

In the case $\alpha = 0$, we get an equivalent norm on modulation spaces $M^s_{p,q}$ (see [6,9,11,16] for the original definition by using the short-time Fourier transform):

$$\|f\|_{M^s_{p,q}} = \left(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{qs} \left\|\mathscr{F}^{-1}\psi_k\mathscr{F}f\right\|^q_{L^p(\mathbb{R}^n)}\right)^{1/q}. \tag{2.5}$$

The modulation spaces $M^s_{p,q}$ in the case $0 < p, q < 1$ were studied in [21–23] by using the norm (2.5). Soon after, Kobayashi [12] independently considered such a generalization in the case $0 < p, q < 1$ (see also [20]).

Recalling that $\bar{x} = (x_1, \dots, x_{n-1})$, we also define the following anisotropic modulation spaces $M^s_{p,q,r}$ for which the norm is defined as

$$\|f\|_{M^s_{p,q,r}} = \left(\sum_{k_n\in\mathbb{Z}} \left(\sum_{\bar{k}\in\mathbb{Z}^{n-1}} \langle\bar{k}\rangle^{qs} \left\|\mathscr{F}^{-1}\psi_k\mathscr{F}f\right\|^q_{L^p(\mathbb{R}^n)}\right)^{r/q}\right)^{1/r}.$$

Similarly, for $\bar{\bar{x}} = (x_1, \dots, x_{n-2})$, one can define the following

$$\|f\|_{M^s_{p,q,r,r}} = \left(\sum_{(k_{n-1},k_n)\in\mathbb{Z}^2} \left(\sum_{\bar{\bar{k}}\in\mathbb{Z}^{n-2}} \langle\bar{\bar{k}}\rangle^{qs} \left\|\mathscr{F}^{-1}\psi_k\mathscr{F}f\right\|^q_{L^p(\mathbb{R}^n)}\right)^{r/q}\right)^{1/r}.$$

The anisotropic versions are of importance for the trace of modulation spaces. Comparing modulation spaces $M^s_{p,q}$ with anisotropic modulation spaces $M^s_{p,q,r}$, we see that $M^s_{p,q}$ is, but $M^s_{p,q,r}$ is not rotational invariant. Using the almost orthogonality of $\{\psi_k\}_{k\in\mathbb{Z}^n}$, we see that the $M^s_{p,q,r}$ is independent of p-BAPU. Moreover, recalling that $\|f\|_{M^s_{p,q,r}}$ is the function sequence $\{\Box_k f\}_{k\in\mathbb{Z}^n}$ equipped with the $\ell^r_{k_n}\ell^q_{\bar{k}}L^p$ norm, it is easy to see that $M^s_{p,q,r}$ is a quasi-Banach space for any $s\in\mathbb{R}$, $p, q, r \in (0, \infty]$ and a Banach space for any $s \in \mathbb{R}$, $1 \leqslant p, q, r \leqslant \infty$. Moreover, the Schwartz space is dense in $M^s_{p,q,r}$ if $p, q, r < \infty$. The proofs are similar to those of modulation spaces, cf. [21]. Since this paper is mainly concentrated the trace properties, the anisotropic modulation spaces will be studied in a subsequent paper.

2.4. Besov spaces

Write $\varphi(\cdot) = \eta(\cdot) - \eta(2\cdot)$ and $\varphi_k := \varphi(2^{-k}\cdot)$ for $k \geqslant 1$. $\varphi_0 := 1 - \sum_{k\geqslant 1}\varphi_k$. For simplicity, we write $\Delta_k = \mathscr{F}^{-1}\varphi_k\mathscr{F}$. The norms on Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ are defined as follow (see [2]):

$$\|f\|_{B^s_{p,q}} = \left(\sum_{j=0}^{\infty} 2^{sjq}\|\Delta_j f\|^q_p\right)^{1/q}.$$

For our purpose, we also need the following

$$\tilde{B}^s_{p,q}(\mathbb{R}^n) = \left(\sum_{k=0}^{\infty} k2^{skq}\|\Delta_k f\|^q_{L^p(\mathbb{R}^n)}\right)^{1/q}.$$

In the case $1 < p < \infty$, using Lizorkin's decomposition of $\mathbb{R}^n$, we have an equivalent quasi-norm on $B^s_{p,q}(\mathbb{R}^n)$. Let

$$K_k = \{x:\ |x_j| < 2^k,\ j = 1, 2, \dots, n\} \setminus \{x:\ |x_j| < 2^{k-1},\ j = 1, 2, \dots, n\},$$

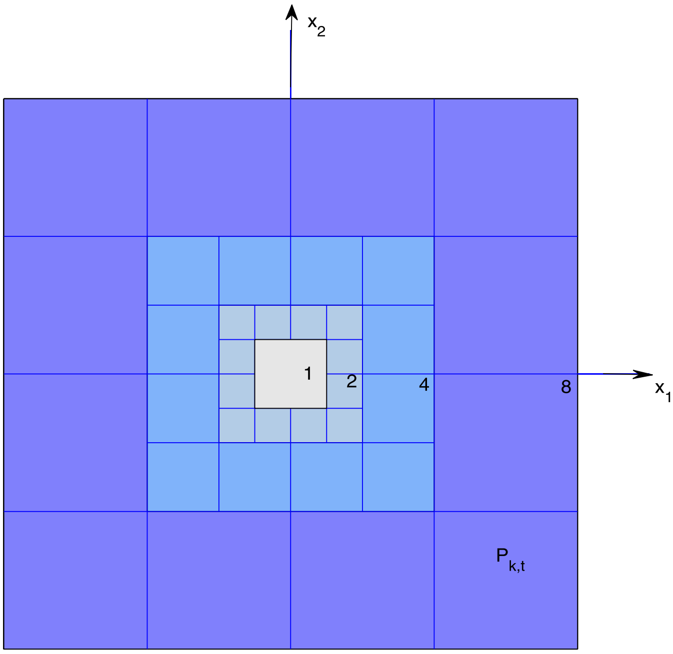

Fig. 2. 1-covering, the case $n=2$.

where $k \in \mathbb{Z}^+$ and

$$K_0 = \{x\colon\ |x_j| \leqslant 1,\ j = 1,2,\dots,n\}.$$

Subdivide K_k with $k = 1,2,3\dots$, by the $3n$ hyper-planes $\{x\colon\ x_m = 0\}$ and $\{x\colon\ x_m = \pm 2^{k-1}\}$, where $m = 1,\dots,n$, into cubes $P_{k,t}$. If k is fixed, we obtain $T = 4^n - 2^n$ cubes. The cubes near the n-th axis are numbered by $t = 1,\dots,2^n$ in an arbitrary way and the others are numbered by $t = 2^n + 1,\dots,T$. Let $P_{0,t} = K_0$, if $t = 1,\dots,T$. Then (see Fig. 2)

$$\mathbb{R}_n = \bigcup_{k=0}^{\infty} \bar{K}_k = \bigcup_{k=0}^{\infty} \bigcup_{t=1}^{T} \bar{P}_{k,t}.$$

Let $\chi_{k,t}$ be a characteristic function on $P_{k,t}$. Then

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)} \sim \left(\sum_{k=0}^{\infty} \sum_{t=1}^{T} 2^{skq} \big\| \mathscr{F}^{-1} \chi_{k,t} \mathscr{F} f \big\|^q_{L^p(\mathbb{R}^n)} \right)^{1/q}.$$

We construct two new norms. For simplicity, we write

$$\Delta_{k,t} = \mathscr{F}^{-1} \chi_{k,t} \mathscr{F}.$$

Define

$$B^{s_1,s_2}_{p,q} = \left(\sum_{k=0}^{\infty} \left(\sum_{t=1}^{2^n} 2^{s_1 kq} \|\Delta_{k,t} f\|^q_{L^p(\mathbb{R}^n)} + \sum_{t=2^n+1}^{T} 2^{s_2 kq} \|\Delta_{k,t} f\|^q_{L^p(\mathbb{R}^n)} \right) \right)^{1/q},$$

$$\tilde{B}^{s_1,s_2}_{p,q} = \left(\sum_{k=0}^{\infty} \left(\sum_{t=1}^{2^n} k 2^{s_1 kq} \|\Delta_{k,t} f\|^q_{L^p(\mathbb{R}^n)} + \sum_{t=2^n+1}^{T} 2^{s_2 kq} \|\Delta_{k,t} f\|^q_{L^p(\mathbb{R}^n)} \right) \right)^{1/q}.$$

3. Main results

We have the following results.

Theorem 3.1. *Let $n \geqslant 2$, $0 < p,q < \infty$, $s \in \mathbb{R}$. Then*

$$\mathbb{T}: f(x) \to f(\bar{x}, 0), \quad \bar{x} = (x_1,\dots,x_{n-1}) \tag{3.1}$$

is a trace-reaction from $M^s_{p,q,\,p\wedge q\wedge 1}(\mathbb{R}^n)$ onto $M^s_{p,q}(\mathbb{R}^{n-1})$.

Similarly, we can show that $\mathbb{T}: f(x) \to f(\bar{\bar{x}}, 0, 0)$ ($\bar{\bar{x}} = (x_1, \dots, x_{n-2})$) is a trace-reaction from $M^s_{p,q,p\wedge q\wedge 1,p\wedge q\wedge 1}(\mathbb{R}^n)$ onto $M^s_{p,q}(\mathbb{R}^{n-2})$. Theorem 3.1 is sharp in the sense that $\mathbb{T}: M^s_{p,q,r}(\mathbb{R}^n) \nrightarrow M^s_{p,q}(\mathbb{R}^{n-1})$ for some $r > 1$, $p, q \geqslant 1$. In view of the basic embedding $M^s_{p,q} \subset M^{s_2}_{p,q,q_2}$ for $s - s_2 > 1/q - 1/q_2 > 0$, $s \geqslant 0$, we immediately have

Corollary 3.2. *Let $n \geqslant 2$, $0 < p, q < \infty$, $s \geqslant 0$. Let $\mathbb{T}$ be as in (3.1). Then for any $\varepsilon > 0$,*

$$\mathbb{T}: M_{p,q}^{s+\frac{1}{p\wedge q\wedge 1}-\frac{1}{q}+\varepsilon}\big(\mathbb{R}^n\big) \to M^s_{p,q}\big(\mathbb{R}^{n-1}\big).$$

One may ask if Corollary 3.2 holds for the limit case $s = \varepsilon = 0$? However, we can give a counterexample to show that $\mathbb{T}: M^{1/q'}_{p,q}(\mathbb{R}^n) \nrightarrow M^0_{p,q}(\mathbb{R}^{n-1})$ in the case $p, q > 1$. Write

$$s_p = (n-1)\big(1/(p\wedge 1) - 1\big).$$

It is easy to see that $s_p = 0$ for $p \geqslant 1$ and $s_p = (n-1)(1/p - 1)$ for $p < 1$. For the trace of α-modulation spaces, we have the following result.

Theorem 3.3. *Let $n \geqslant 2$, $0 < p, q < \infty$, $s \geqslant \alpha(n-1)/q + \alpha s_p$. Let $\mathbb{T}$ be as in (3.1). Then*

$$\mathbb{T}: M^{s+\alpha/p,\alpha}_{p,p\wedge q\wedge 1}\big(\mathbb{R}^n\big) \to M^{s,\alpha}_{p,q}\big(\mathbb{R}^{n-1}\big).$$

The case $s < \alpha(n-1)/q + \alpha s_p$ is more complicated. We have the following

Theorem 3.4. *Let $n \geqslant 2$, $0 < p, q < \infty$, $s < \alpha(n-1)/q + \alpha s_p$. Let $\mathbb{T}$ be as in (3.1). Then*

$$\mathbb{T}: M^{s+\sigma_{\alpha,p,q},\alpha}_{p,p\wedge q\wedge 1}\big(\mathbb{R}^n\big) \to M^{s,\alpha}_{p,q}\big(\mathbb{R}^{n-1}\big),$$

where

$$\sigma_{\alpha,p,q} = \begin{cases} \alpha/p + (1-\alpha)[\alpha(n-1)/q + \alpha s_p - s], & qs + (n-1)(1-\alpha) - q\alpha s_p > 0, \\ \alpha/p + \alpha s_p - s + \varepsilon, & qs + (n-1)(1-\alpha) - q\alpha s_p = 0, \\ \alpha/p + \alpha s_p - s, & qs + (n-1)(1-\alpha) - q\alpha s_p < 0. \end{cases}$$

Theorem 3.3 is sharp in the case $s \geqslant 0$, $p = q = 1$. As the end of this paper, we consider the trace of Besov spaces. If $s > s_p$, the corresponding result has been obtained in [8]. If $s \leqslant s_p$, we have the following trace theorem for Besov spaces:

Theorem 3.5. *Let $n \geqslant 2$, $0 < p, q < \infty$, $s \leqslant s_p$. Let $\mathbb{T}$ be as in (3.1). Then we have*

$$\mathbb{T}: \tilde{B}^{s_p+1/p}_{p,p\wedge q\wedge 1}\big(\mathbb{R}^n\big) \to B^{s_p}_{p,q}\big(\mathbb{R}^{n-1}\big),$$

and

$$\mathbb{T}: B^{s_p+1/p}_{p,p\wedge q\wedge 1}\big(\mathbb{R}^n\big) \to B^{s}_{p,q}\big(\mathbb{R}^{n-1}\big)$$

in the case $s < s_p$. Moreover, when $1 < p < \infty$, we have

$$\mathbb{T}: \tilde{B}^{1/p,1/p}_{p,q\wedge 1}\big(\mathbb{R}^n\big) \to B^{0}_{p,q}\big(\mathbb{R}^{n-1}\big),$$
$$\mathbb{T}: B^{1/p,s+1/p}_{p,q\wedge 1}\big(\mathbb{R}^n\big) \to B^{s}_{p,q}\big(\mathbb{R}^{n-1}\big), \quad s < 0.$$

4. Proof of Theorem 3.1

If there is no explanation, we always assume $r = 1/2$ in the p-BAPU for the case of modulation spaces. First, we indicate our idea in the proof. Since the Schwartz space is dense in any (anisotropic) modulation spaces if the space-indices p, q, r are not infinity, it suffices to show the trace theorems for Schwartz functions. Recall that $\bar{x} = (x_1, \dots, x_{n-1})$. To establish the relation of $f(x)$ and $f(\bar{x}, 0)$, an important observation is that

$$\big(\mathscr{F}^{-1}_{\bar{\xi}}\psi_{\bar{k}}\mathscr{F}_{\bar{x}} f\big)(\bar{x}, 0) = \sum_{l\in\mathbb{Z}^n,\, |\bar{k}-\bar{l}|\leqslant C} \big(\mathscr{F}^{-1}_{\bar{\xi}}\psi_{\bar{k}}\big) * \big(\mathscr{F}^{-1}\psi_l\mathscr{F} f\big)(\bar{x}, 0), \tag{4.1}$$

where $\psi_{\bar{k}}$ is as in (4.6). So, if (4.1) is taken any sequence norm on $\bar{k}$, the right-hand side is a finite summation on $\bar{l}$. One can estimate $(\mathscr{F}^{-1}\psi_l\mathscr{F} f)(\bar{x}, 0)$ in $L^p(\mathbb{R}^{n-1})$ by a maximal function, see (4.5) below. Finally, one needs to distinguish the cases $p < 1$ and $p \geqslant 1$ by different techniques.

Lemma 4.1. *(See Triebel [19].) Let Ω be a compact subset of $\mathbb{R}^n$ and $0 < p \leqslant \infty$. Denote $L_p^{\Omega} = \{f \in L^p\colon \operatorname{supp} \hat{f} \subset \Omega\}$. Let $0 < r < p$. Then*

$$\left\| \sup_{z\in\mathbb{R}^n} \frac{f(\cdot - z)}{1+|z|^{n/r}} \right\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)},$$

holds for any $f \in L_p^{\Omega}$.

Assume that $\operatorname{Supp} \hat{f} \subset B(\xi_0, R)$. It is easy to see that for $g = e^{ix\xi_0} f(R^{-1}\cdot)$, $\hat{g} = R^n \hat{f}(R(\xi - \xi_0))$. It follows that $\operatorname{supp} \hat{g} \subset B(0,1)$. Taking $\Omega = B(0,1)$ in Lemma 4.1, we find that

$$\left\| \sup_{z\in\mathbb{R}^n} \frac{g(\cdot - z)}{1+|z|^{n/r}} \right\|_{L^p(\mathbb{R}^n)} \lesssim \|g\|_{L^p(\mathbb{R}^n)}.$$

By scaling, we have

$$\left\| \sup_{z\in\mathbb{R}^n} \frac{f(\cdot - z)}{1+|Rz|^{n/r}} \right\|_{L^p(\mathbb{R}^n)} \leqslant C\|f\|_{L^p(\mathbb{R}^n)}. \tag{4.2}$$

Note that the constant C in (4.2) is independent of $f \in L^p_{B(\xi_0,R)} = \{f \in L^p\colon \operatorname{supp} \hat{f} \subset B(\xi_0, R)\}$. It is also independent of $\xi_0 \in \mathbb{R}^n$.

For convenience, we write

$$\Box_k = \mathscr{F}^{-1}\psi_k\mathscr{F}, \quad k \in \mathbb{Z}^n.$$

We define the maximum function $M_k^* f$ as follows:

$$M_k^* f = \sup_{y\in\mathbb{Z}^n} \frac{|\Box_k f(x-y)|}{1+|y|^{n/r}}. \tag{4.3}$$

Taking $y_1 = \cdots = y_{n-1} = 0$, $y_n = x_n$ in (4.3), we have for $|x_n| \leqslant 1$,

$$\left|(\Box_k f)(\bar{x}, 0)\right| \lesssim \left|M_k^* f(x)\right|, \quad \bar{x} = (x_1, \ldots, x_{n-1}).$$

Hence

$$\left\|(\Box_k f)(\cdot, 0)\right\|_{L^p(\mathbb{R}^{n-1})} \lesssim \left\|M_k^* f(\cdot, x_n)\right\|_{L^p(\mathbb{R}^{n-1})}. \tag{4.4}$$

Integrating (4.4) over $x_n \in [0,1]$, one has that

$$\left\|(\Box_k f)(\cdot, 0)\right\|^p_{L^p(\mathbb{R}^{n-1})} \lesssim \int_{\mathbb{R}} \left\|M_k^* f(\cdot, x_n)\right\|^p_{L^p(\mathbb{R}^{n-1})} dx_n.$$

Hence

$$\left\|(\Box_k f)(\cdot, 0)\right\|_{L^p(\mathbb{R}^{n-1})} \lesssim \left\|M_k^* f\right\|_{L^p(\mathbb{R}^n)}. \tag{4.5}$$

We denote by $\mathscr{F}_{\bar{x}}$ ($\mathscr{F}^{-1}_{\bar{\xi}}$) the partial (inverse) Fourier transform on $\bar{x}$ ($\bar{\xi}$). Write $\psi_{\bar{k}}(\bar{x})$ as the p-BAPU functions in $\mathbb{R}^{n-1}$ as in (2.3), i.e.,

$$\psi_{\bar{k}}(\bar{\xi}) = \frac{\phi_{k_1}(\xi_1)\ldots\phi_{k_{n-1}}(\xi_{n-1})}{\sum_{\bar{k}\in\mathbb{Z}^{(n-1)}} \phi_{k_1}(\xi_1)\cdots\phi_{k_{n-1}}(\xi_{n-1})}, \quad \bar{k} \in \mathbb{Z}^{(n-1)}. \tag{4.6}$$

Then we have

$$\left(\mathscr{F}^{-1}_{\bar{\xi}}\psi_{\bar{k}}\mathscr{F}_{\bar{x}} f\right)(\bar{x}, 0) = \sum_{l\in\mathbb{Z}^n} \left(\mathscr{F}^{-1}_{\bar{\xi}}\psi_{\bar{k}}\mathscr{F}_{\bar{x}}\mathscr{F}^{-1}\psi_l\mathscr{F} f\right)(\bar{x}, 0) = \sum_{l\in\mathbb{Z}^n} \left(\mathscr{F}^{-1}_{\bar{\xi}}\psi_{\bar{k}}\right) * \left(\mathscr{F}^{-1}\psi_l\mathscr{F} f\right)(\bar{x}, 0).$$

From the support property of ψ_l as in (2.3), we find that

$$\psi_{\bar{k}}\psi_l = 0, \quad \text{if } |\bar{l} - \bar{k}| \geqslant C.$$

Hence

$$\left(\mathscr{F}^{-1}_{\bar{\xi}}\psi_{\bar{k}}\mathscr{F}_{\bar{x}} f\right)(\bar{x}, 0) = \sum_{l\in\mathbb{Z}^n,\, |\bar{k}-\bar{l}|\leqslant C} \left(\mathscr{F}^{-1}_{\bar{\xi}}\psi_{\bar{k}}\right) * \left(\left(\mathscr{F}^{-1}\psi_l\mathscr{F} f\right)(\cdot, 0)\right).$$

Case 1. $1 \leqslant p < \infty$. Using Young's inequality, (4.2) and (4.5), we obtain

$$\begin{aligned}\left\|\mathscr{F}_{\bar{\xi}}^{-1}\psi_{\bar{k}}\mathscr{F}_{\bar{x}}f(\bar{x},0)\right\|_{L^p(\mathbb{R}^{n-1})} &\lesssim \sum_{l\in\mathbb{Z}^n,\,|\bar{k}-\bar{l}|\leqslant C}\left\|\mathscr{F}_{\bar{\xi}}^{-1}\psi_{\bar{k}}\right\|_{L^1(\mathbb{R}^{n-1})}\left\|\mathscr{F}^{-1}\psi_l\mathscr{F}f\right\|_{L^p(\mathbb{R}^{n-1})}\\ &\lesssim \sum_{l\in\mathbb{Z}^n,\,|\bar{k}-\bar{l}|\leqslant C}\left\|M_l^*f\right\|_{L^p(\mathbb{R}^n)}\\ &\lesssim \sum_{l\in\mathbb{Z}^n,\,|\bar{k}-\bar{l}|\leqslant C}\|\Box_l f\|_{L^p(\mathbb{R}^n)}.\end{aligned}$$

Hence,

$$\left\|f(\bar{x},0)\right\|_{M^s_{p,q}(\mathbb{R}^{n-1})} \lesssim \left(\sum_{\bar{k}\in\mathbb{Z}^{n-1}}\langle\bar{k}\rangle^{sq}\left(\sum_{l\in\mathbb{Z}^n,\,|\bar{k}-\bar{l}|\leqslant C}\|\Box_l f\|_{L^p(\mathbb{R}^n)}\right)^q\right)^{1/q}.$$

If $0 < q \leqslant 1$, then

$$\begin{aligned}\left\|f(\bar{x},0)\right\|_{M^s_{p,q}(\mathbb{R}^{n-1})} &\lesssim \left(\sum_{l\in\mathbb{Z}^n}\sum_{\bar{k}\in\mathbb{Z}^{n-1}}\langle\bar{k}\rangle^{sq}\chi_{(|\bar{k}-\bar{l}|\leqslant C)}\|\Box_l f\|^q_{L^p(\mathbb{R}^n)}\right)^{1/q}\\ &\lesssim \left(\sum_{l\in\mathbb{Z}^n}\langle\bar{l}\rangle^{sq}\|\Box_l f\|^q_{L^p(\mathbb{R}^n)}\right)^{1/q} = \|f\|_{M^s_{p,q,q}}.\end{aligned}$$

If $1 \leqslant q < \infty$, using Minkowski's inequality together with Hölder's inequality,

$$\begin{aligned}\left\|f(\cdot,0)\right\|_{M^s_{p,q}(\mathbb{R}^{n-1})} &\lesssim \left(\sum_{\bar{k},\bar{l}\in\mathbb{Z}^{n-1}}\langle\bar{k}\rangle^{sq}\left(\sum_{l_n\in\mathbb{Z}}\chi_{(|\bar{k}-\bar{l}|\leqslant C)}\|\Box_l f\|_{L^p(\mathbb{R}^n)}\right)^q\right)^{1/q}\\ &\lesssim \sum_{l_n\in\mathbb{Z}}\left(\sum_{\bar{k},\bar{l}\in\mathbb{Z}^{n-1}}\langle\bar{k}\rangle^{sq}\chi_{(|\bar{k}-\bar{l}|\leqslant C)}\|\Box_l f\|^q_{L^p(\mathbb{R}^n)}\right)^{1/q}\\ &\lesssim \sum_{l_n\in\mathbb{Z}}\left(\sum_{\bar{l}\in\mathbb{Z}^{n-1}}\langle\bar{l}\rangle^{sq}\|\Box_l f\|^q_{L^p(\mathbb{R}^n)}\right)^{1/q}\\ &= \|f\|_{M^s_{p,q,1}}.\end{aligned}$$

To begin with the proof for the case $0 < p < 1$, we need the following lemma:

Lemma 4.2. *Let $0 < p \leqslant 1$. Suppose that $f, g \in L^p_{B(x_0,R)}$, then there exists a constant $C > 0$ which is independent of $x_0 \in \mathbb{R}^n$ and $R > 0$ such that*

$$\|f * g\|_p \leqslant C R^{n(\frac{1}{p}-1)}\|f\|_p\|g\|_p.$$

Proof. In the case $f, g \in L^p_{B(0,1)}$, we have

$$\|f * g\|_p \lesssim \|f\|_p\|g\|_p.$$

Taking $f_\lambda = f(\lambda\cdot)$ and $g_\lambda = g(\lambda\cdot)$, we see that

$$f_{R^{-1}}, g_{R^{-1}} \in L^p_{B(0,1)}, \quad \text{if } f, g \in L^p_{B(0,R)}.$$

Hence, for any $f, g \in L^p_{B(0,R)}$,

$$\|f_{R^{-1}} * g_{R^{-1}}\|_p \lesssim \|f_{R^{-1}}\|_p\|g_{R^{-1}}\|_p.$$

By scaling, we have

$$\|f * g\|_p \lesssim R^{n(\frac{1}{p}-1)}\|f\|_p\|g\|_p.$$

By a translation $\widehat{e^{ix_0x}f} = \hat{f}(\xi - x_0)$, we immediately have the result, as desired.

Case 2. $0 < p < 1$. By Lemma 4.2, (4.2) and (4.5),

$$\begin{aligned}\big\|\mathscr{F}_{\bar\xi}^{-1}\psi_{\bar k}\mathscr{F}_{\bar x} f(\bar x,0)\big\|_{L^p(\mathbb{R}^{n-1})}^p &\lesssim \sum_{l\in\mathbb{Z}^n,\,|\bar k-\bar l|\leqslant C}\big\|\mathscr{F}_{\bar\xi}^{-1}\psi_{\bar k}\big\|_{L^p(\mathbb{R}^{n-1})}^p\big\|\big(\mathscr{F}^{-1}\psi_l\mathscr{F} f\big)(\cdot,0)\big\|_{L^p(\mathbb{R}^{n-1})}^p\\ &\lesssim \sum_{l\in\mathbb{Z}^n,\,|\bar k-\bar l|\leqslant C}\big\|\big(\mathscr{F}^{-1}\psi_l\mathscr{F} f\big)(\cdot,0)\big\|_{L^p(\mathbb{R}^{n-1})}^p\\ &\lesssim \sum_{l\in\mathbb{Z}^n,\,|\bar k-\bar l|\leqslant C}\|\Box_l f\|_{L^p(\mathbb{R}^n)}^p.\end{aligned}$$

It follows that

$$\big\|f(\bar x,0)\big\|_{M_{p,q}^{s,\alpha}} \lesssim \bigg(\sum_{\bar k\in\mathbb{Z}^n}\langle\bar k\rangle^{sq}\bigg(\sum_{l\in\mathbb{Z}^n}\|\Box_l f\|_{L^p(\mathbb{R}^n)}^p\chi_{(|\bar k-\bar l|\leqslant C)}\bigg)^{q/p}\bigg)^{1/q}.$$

If $q \leqslant p$, one has that

$$\begin{aligned}\big\|f(\bar x,0)\big\|_{M_{p,q}^{s}(\mathbb{R}^{n-1})} &\lesssim \bigg(\sum_{l\in\mathbb{Z}^n}\sum_{\bar k\in\mathbb{Z}^n}\langle\bar k\rangle^{sq}\big\|\Box_l^\alpha f\big\|_{L^p(\mathbb{R}^n)}^q\chi_{(|\bar k-\bar l|\leqslant C)}\bigg)^{1/q}\\ &\lesssim \|f\|_{M_{p,q}^s}.\end{aligned}$$

If $q \geqslant p$, using Minkowski's inequality, we have

$$\begin{aligned}\big\|f(\bar x,0)\big\|_{M_{p,q}^{s}(\mathbb{R}^{n-1})} &\lesssim \bigg(\sum_{l_n\in\mathbb{Z}}\bigg(\sum_{\bar k,\bar l\in\mathbb{Z}^{n-1}}\langle\bar k\rangle^{sq}\|\Box_l f\|_{L^p(\mathbb{R}^n)}^q\chi_{(|\bar k-\bar l|\leqslant C)}\bigg)^{p/q}\bigg)^{1/p}\\ &\lesssim \bigg(\sum_{l_n\in\mathbb{Z}}\bigg(\sum_{\bar l\in\mathbb{Z}^{n-1}}\langle\bar l\rangle^{sq}\|\Box_l f\|_{L^p(\mathbb{R}^n)}^q\bigg)^{p/q}\bigg)^{1/p}\\ &= \|f\|_{M_{p,q,p}^s}.\end{aligned}$$

In order to show $\mathbb{T}$ is a trace-reaction, we need to show the existence of $\mathbb{T}^{-1}$. Let η be as in (2.2) satisfying $(\mathscr{F}_{\xi_n}^{-1}\eta)(0) = 1$. For any $f \in M_{p,q}^s(\mathbb{R}^{n-1})$, we define

$$g(x) = \big[\big(\mathscr{F}_{\xi_n}^{-1}\eta\big)(x_n)\big]f(\bar x) := \big(\mathbb{T}^{-1}f\big)(x).$$

It is easy to see that $g(\bar x,0) = f(\bar x)$ and $\Box_k g = 0$ for $|k_n| \geqslant 3$. Hence,

$$\begin{aligned}\|g\|_{M_{p,q,p\wedge q\wedge 1}^s(\mathbb{R}^n)} &\lesssim \bigg(\sum_{k_n\in\mathbb{Z}}\bigg(\sum_{\bar k\in\mathbb{Z}^{n-1}}\langle\bar k\rangle^{sq}\|\Box_k g\|_{L^p(\mathbb{R}^n)}^q\bigg)^{p\wedge q\wedge 1/q}\bigg)^{1/p\wedge q\wedge 1}\\ &= \sum_{|k_n|\leqslant 2}\bigg(\sum_{\bar k\in\mathbb{Z}^{n-1}}\langle\bar k\rangle^{sq}\|\Box_{\bar k} f\|_{L^p(\mathbb{R}^n)}^q\big\|\mathscr{F}_{\xi_n}^{-1}\eta\big\|_{L^p(\mathbb{R})}^q\bigg)^{1/q}\\ &\lesssim \|f\|_{M_{p,q}^s(\mathbb{R}^{n-1})}.\end{aligned}$$

It follows that $\mathbb{T}^{-1} : M_{p,q}^s(\mathbb{R}^{n-1}) \to M_{p,q,p\wedge q\wedge 1}^s(\mathbb{R}^n)$. $\Box$

As the end of this section, we show that Theorem 3.1 and Corollary 3.2 are sharp conclusions. First, we show that $\mathbb{T} : M_{p,q,r}^0(\mathbb{R}^n) \nrightarrow M_{p,q}^0(\mathbb{R}^{n-1})$ if $r > 1$. Let η be as in (2.2), $f = \mathscr{F}^{-1}(\eta(2\xi_1)\ldots\eta(2\xi_n))$. For $k = (k_1,\ldots,k_n)$, we denote

$$F(x) = \sum_{|k_n|\leqslant 2^N}\langle k_n\rangle^{-1}e^{\mathrm{i}k_nx_n}f(x).$$

It is easy to see that

$$\mathscr{F}F(\xi) = \sum_{|k_n|\leqslant 2^N}\langle k_n\rangle^{-1}\eta(2\xi_1)\ldots\eta\big(2(\xi_n-k_n)\big).$$

Hence, $\Box_k F = 0$ if $\max_{i=1,\ldots,n-1}|k_i| > 2$ or $|k_n| > 2^N + 1$. In view of the definition

$$\|F\|_{M^0_{p,q,r}(\mathbb{R}^n)} \lesssim \left(\sum_{|k_n|\leqslant 2^N+1}\left(\sum_{|k_i|\leqslant 2,\,1\leqslant i\leqslant n-1}\|\Box_k F\|^q_{L^p(\mathbb{R}^n)}\right)^{r/q}\right)^{1/r}$$
$$\lesssim \sum_{|k_i|\leqslant 2,\,1\leqslant i\leqslant n-1}\left(\sum_{|k_n|\leqslant 2^N+1}\|\Box_k F\|^r_{L^p(\mathbb{R}^n)}\right)^{1/r}$$
$$\lesssim \left(\sum_{|k_n|\leqslant 2^N+1}\langle k_n\rangle^{-r}\right)^{1/r} \lesssim 1.$$

On the other hand, we may assume that $(\mathscr{F}^{-1}_{\xi_n}\eta(2\cdot))(0)=1$. We have

$$F(\bar{x},0)=\left(\sum_{|k_n|\leqslant 2^N}\langle k_n\rangle^{-1}\right)\mathscr{F}^{-1}_{\bar{\xi}}\big[\eta(2\xi_1)\dots\eta\big(2(\xi_{n-1})\big)\big].$$

So,

$$\|F\|_{M^0_{p,q}(\mathbb{R}^{n-1})} \gtrsim \left(\sum_{|k_n|\leqslant 2^N}\langle k_n\rangle^{-1}\right)\left(\sum_{|k_i|\leqslant 2,\,1\leqslant i\leqslant n-1}\big\|\mathscr{F}^{-1}\psi_{\bar{k}}\eta(2\xi_1)\dots\eta(2\xi_{n-1})\big\|^q_{L^p(\mathbb{R}^{n-1})}\right)^{1/q}$$
$$\gtrsim N.$$

Let $N\to\infty$, we have $\mathbb{T}: M^0_{p,q,r}(\mathbb{R}^n)\nrightarrow M^0_{p,q}(\mathbb{R}^{n-1})$.

Next, we show that $\mathbb{T}: M^{1/q'}_{p,q}(\mathbb{R}^n)\nrightarrow M^0_{p,q}(\mathbb{R}^{n-1})$ as $q>1$. For $k=(k_1,\dots,k_n)$, we denote

$$F(x)=\sum_{|k_n|\leqslant 2^N}\frac{1}{\langle k_n\rangle\ln\langle k_n\rangle}e^{\mathrm{i}k_nx_n}f(x).$$

Similarly as in the above, we have

$$\|F\|_{M^{1/q'}_{p,q}(\mathbb{R}^n)} \lesssim \left(\sum_{|k_n|\leqslant 2^N+1}\ \sum_{|k_i|\leqslant 2,\,1\leqslant i\leqslant n-1}\langle k_n\rangle^{q-1}\|\Box_k F\|^q_{L^p(\mathbb{R}^n)}\right)^{1/q}$$
$$\lesssim \left(\sum_{|k_n|\leqslant 2^N+1}\frac{1}{\langle k_n\rangle\ln^q\langle k_n\rangle}\right)^{1/q}\lesssim 1.$$

On the other hand,

$$\|F\|_{M^0_{p,q}(\mathbb{R}^{n-1})} \gtrsim \left(\sum_{|k_n|\leqslant 2^N}\frac{1}{\langle k_n\rangle\ln\langle k_n\rangle}\right)\to\infty,\quad N\to\infty.$$

5. Proofs of Theorems 3.3 and 3.4

For convenience, we write

$$\Box^\alpha_{k,j}=\mathscr{F}^{-1}\psi_{k,j}\mathscr{F},\quad k\in\mathscr{K}_j,\ j\in\mathbb{Z}_+.$$

We define the maximum function $M^*_{k,j}f$ as follows:

$$M^*_{k,j}f=\sup_{y\in\mathbb{Z}^n}\frac{|\Box^\alpha_{k,j}f(x-y)|}{1+|\langle j\rangle^{\alpha/(1-\alpha)}y|^{n/r}}. \tag{5.1}$$

Taking $y_1=\dots=y_{n-1}=0$, $y_n=x_n$ in (5.1), we have for $\langle j\rangle^{-\alpha/(1-\alpha)}\leqslant|x_n|\leqslant 2\langle j\rangle^{-\alpha/(1-\alpha)}$,

$$\big|\big(\Box^\alpha_{k,j}f\big)(\bar{x},0)\big|\lesssim\big|M^*_{k,j}f(x)\big|,\quad \bar{x}=(x_1,\dots,x_{n-1}).$$

Hence

$$\big\|\big(\Box^\alpha_{k,j}f\big)(\cdot,0)\big\|_{L^p(\mathbb{R}^{n-1})}\lesssim\big\|M^*_{k,j}f(\cdot,x_n)\big\|_{L^p(\mathbb{R}^{n-1})}. \tag{5.2}$$

Integrating (5.2) over $x_n\in[\langle j\rangle^{-\alpha/(1-\alpha)},2\langle j\rangle^{-\alpha/(1-\alpha)}]$, one has that

$$\left\|\left(\Box_{k,j}^{\alpha} f\right)(\cdot,0)\right\|_{L^p(\mathbb{R}^{n-1})}^p \lesssim \langle j\rangle^{\alpha/(1-\alpha)} \int_{\mathbb{R}} \left\|M_{k,j}^* f(\cdot,x_n)\right\|_{L^p(\mathbb{R}^{n-1})}^p dx_n.$$

Hence

$$\left\|\left(\Box_{k,j}^{\alpha} f\right)(\cdot,0)\right\|_{L^p(\mathbb{R}^{n-1})} \lesssim \langle j\rangle^{\alpha/p(1-\alpha)} \left\|M_{k,j}^* f\right\|_{L^p(\mathbb{R}^n)}. \tag{5.3}$$

We denote by $\mathscr{F}_{\bar{x}}(\mathscr{F}_{\bar{\xi}}^{-1})$ the partial (inverse) Fourier transform on $\bar{x}=(x_1,\ldots,x_{n-1})$ ($\bar{\xi}=(\xi_1,\ldots,\xi_{n-1})$). Write $\psi_{m,l}(\bar{x})$ as the p-BAPU functions in $\mathbb{R}^{n-1}$ as in (2.4). Noticing that the Schwartz space is dense in α-modulation $M_{p,q}^{s,\alpha}$ for $p,q<\infty$, it suffices to treat f as a Schwartz function. So, by the definition,

$$\|f\|_{M_{p,q}^{s,\alpha}(\mathbb{R}^{n-1})} = \left(\sum_{l\in\mathbb{Z}_+}\sum_{m\in\mathscr{K}_l^{n-1}} \langle l\rangle^{\frac{qs}{1-\alpha}} \left\|\left(\mathscr{F}_{\bar{\xi}}^{-1}\psi_{m,l}(\bar{\xi})\mathscr{F}_{\bar{x}} f\right)(\cdot,0)\right\|_{L^p(\mathbb{R}^{n-1})}^q\right)^{1/q}. \tag{5.4}$$

In order to have no confusion, we always denote by $\psi_{m,l}$ the p-BAPU function in $\mathbb{R}^{n-1}$ and by $\psi_{k,j}$ the p-BAPU function in $\mathbb{R}^n$. From the support property of $\psi_{k,j}$, we find that

$$\left(\mathscr{F}_{\bar{\xi}}^{-1}\psi_{m,l}\mathscr{F}_{\bar{x}} f\right)(\bar{x},0) = \sum_{j\geqslant l-C,\, k\in\mathscr{K}_j^n} \left(\mathscr{F}_{\bar{\xi}}^{-1}\psi_{m,l}\mathscr{F}_{\bar{x}}\mathscr{F}^{-1}\psi_{k,j}\mathscr{F} f\right)(\bar{x},0). \tag{5.5}$$

For our purpose we further decompose $\mathscr{K}_j^n$. Denote

$$\mathscr{K}_{j,\lambda}^n = \left\{k\in\mathscr{K}_j^n\colon \max_{1\leqslant i\leqslant n-1}|k_i| = \lambda\right\}, \quad \lambda = r_{j0}, r_{j1},\ldots,r_{jN_j},\ r_{j0}=0.$$

We easily see that $\sum_{k\in\mathscr{K}_j^n} = \sum_{\lambda=0,r_{j1},\ldots,r_{jN_j}}\sum_{\mathscr{K}_{j,\lambda}^n}$, $N_j\sim\langle j\rangle$. Now we divide our discussion into the following four cases.

Case 1. $1\leqslant p<\infty$ and $0<q\leqslant 1$. By (5.4) and (5.5),

$$\begin{aligned}\left\|f(\cdot,0)\right\|_{M_{p,q}^{s,\alpha}(\mathbb{R}^{n-1})}^q \lesssim& \sum_{j\in\mathbb{Z}_+}\sum_{\lambda=0,r_{j1},\ldots,r_{jN_j}}\sum_{k\in\mathscr{K}_{j,\lambda}^n}\sum_{l\leqslant j+C}\sum_{m\in\mathscr{K}_l^{n-1}} \langle l\rangle^{\frac{sq}{1-\alpha}} \\ &\times \left\|\left(\mathscr{F}_{\bar{\xi}}^{-1}\psi_{m,l}\right) * \left(\mathscr{F}^{-1}\psi_{k,j}\mathscr{F} f\right)(\cdot,0)\right\|_{L^p(\mathbb{R}^{n-1})}^q.\end{aligned} \tag{5.6}$$

In order to control (5.6) by $\|f\|_{M_{p,q}^{s+\alpha/p,\alpha}}$, we need to bound the sum $\sum_{l\leqslant j+C}\sum_{m\in\mathscr{K}_l^{n-1}}$. It is easy to see that for fixed k, j,

$$\#\left\{m\in\mathscr{K}_l^{n-1}\colon \operatorname{supp}\psi_{m,l}\cap\operatorname{supp}\psi_{k,j}(\cdot,0)\neq\varnothing\right\} \lesssim \min\left(\langle l\rangle^{n-2},\ \langle j\rangle^{\frac{\alpha(n-2)}{1-\alpha}}/\langle l\rangle^{\frac{\alpha(n-2)}{1-\alpha}}\right). \tag{5.7}$$

Moreover, $k\in\mathscr{K}_{j,r_{ja}}^n$ means that $\operatorname{supp}\psi_{m,l}\cap\operatorname{supp}\psi_{k,j}(\cdot,0)\neq\varnothing$ only if $a^{1-\alpha}\langle j\rangle^\alpha \lesssim l \lesssim (1+a)^{1-\alpha}\langle j\rangle^\alpha$. Hence, in view of Young's inequality, (5.3),

$$\begin{aligned}\Delta_{ja} :=& \sum_{k\in\mathscr{K}_{j,r_{ja}}^n}\sum_{m\in\mathscr{K}_l^{n-1},\, l\leqslant j+C} \langle l\rangle^{\frac{sq}{1-\alpha}} \left\|\left(\mathscr{F}_{\bar{\xi}}^{-1}\psi_{m,l}\right) * \left(\mathscr{F}^{-1}\psi_{k,j}\mathscr{F}\right) f(\cdot,0)\right\|_{L^p(\mathbb{R}^{n-1})}^q \\ \lesssim& \sum_{k\in\mathscr{K}_{j,r_{ja}}^n}\sum_{a^{1-\alpha}\langle j\rangle^\alpha\lesssim l\lesssim(1+a)^{1-\alpha}\langle j\rangle^\alpha} \langle l\rangle^{\frac{sq}{1-\alpha}} \min\left(\langle l\rangle^{n-2},\ \langle j\rangle^{\frac{\alpha(n-2)}{1-\alpha}}/\langle l\rangle^{\frac{\alpha(n-2)}{1-\alpha}}\right) \left\|\left(\mathscr{F}^{-1}\psi_{k,j}\mathscr{F}\right) f(\cdot,0)\right\|_{L^p(\mathbb{R}^{n-1})}^q \\ \lesssim& \sum_{k\in\mathscr{K}_{j,r_{ja}}^n}\sum_{a^{1-\alpha}\langle j\rangle^\alpha\lesssim l\lesssim(1+a)^{1-\alpha}\langle j\rangle^\alpha} \langle l\rangle^{\frac{sq}{1-\alpha}} \min\left(\langle l\rangle^{n-2},\ \langle j\rangle^{\frac{\alpha(n-2)}{1-\alpha}}/\langle l\rangle^{\frac{\alpha(n-2)}{1-\alpha}}\right) \langle j\rangle^{\frac{q\alpha}{p(1-\alpha)}} \left\|\Box_{k,j}^\alpha f\right\|_{L^p(\mathbb{R}^n)}^q.\end{aligned} \tag{5.8}$$

We discuss the following four subcases.

Case 1A. $\alpha(n-1)\leqslant qs$. If $a=0$, one has that

$$\begin{aligned}\Delta_{ja} \lesssim& \sum_{k\in\mathscr{K}_{j,r_{ja}}^n}\sum_{0\leqslant l\lesssim\langle j\rangle^\alpha} \langle l\rangle^{\frac{sq}{1-\alpha}+(n-2)} \langle j\rangle^{\frac{q\alpha}{p(1-\alpha)}} \left\|\Box_{k,j}^\alpha f\right\|_{L^p(\mathbb{R}^n)}^q \\ \lesssim& \sum_{k\in\mathscr{K}_{j,r_{ja}}^n} \langle j\rangle^{\frac{q}{1-\alpha}(\alpha s+\frac{\alpha}{p}+(1-\alpha)\frac{\alpha(n-1)}{q})} \left\|\Box_{k,j}^\alpha f\right\|_{L^p(\mathbb{R}^n)}^q \\ \lesssim& \sum_{k\in\mathscr{K}_{j,r_{ja}}^n} \langle j\rangle^{\frac{q}{1-\alpha}(s+\frac{\alpha}{p})} \left\|\Box_{k,j}^\alpha f\right\|_{L^p(\mathbb{R}^n)}^q.\end{aligned} \tag{5.9}$$

If $a \geqslant 1$, we have

$$\Delta_{ja} \lesssim \sum_{k\in\mathscr{K}^n_{j,r_{ja}}} \sum_{a^{1-\alpha}\langle j\rangle^{\alpha}\lesssim l\lesssim(1+a)^{1-\alpha}\langle j\rangle^{\alpha}} \langle l\rangle^{\frac{sq}{1-\alpha}-\frac{\alpha(n-2)}{1-\alpha}} \langle j\rangle^{\frac{q\alpha}{p(1-\alpha)}+\frac{\alpha(n-2)}{1-\alpha}} \big\|\Box^{\alpha}_{k,j} f\big\|^q_{L^p(\mathbb{R}^n)}$$
$$\lesssim \sum_{k\in\mathscr{K}^n_{j,r_{ja}}} a^{qs-\alpha(n-1)} \langle j\rangle^{\frac{\alpha sq}{1-\alpha}+\frac{q\alpha}{p(1-\alpha)}+\alpha(n-1)} \big\|\Box^{\alpha}_{k,j} f\big\|^q_{L^p(\mathbb{R}^n)}. \tag{5.10}$$

It follows from $qs \geqslant \alpha(n-1)$ that Δ_{ja} takes the maximal value as $a = N_j \sim \langle j\rangle$. Hence,

$$\Delta_{ja} \lesssim \sum_{k\in\mathscr{K}^n_{j,r_{ja}}} \langle j\rangle^{\frac{q}{1-\alpha}(s+\frac{\alpha}{p})} \big\|\Box^{\alpha}_{k,j} f\big\|^q_{L^p(\mathbb{R}^n)}. \tag{5.11}$$

Inserting the estimates of Δ_{ja} as in (5.9) and (5.11) into (5.6), we have

$$\big\|f(\cdot,0)\big\|^q_{M^{s,\alpha}_{p,q}(\mathbb{R}^{n-1})} \lesssim \sum_{j\in\mathbb{Z}_+} \sum_{k\in\mathscr{K}^n_j} \langle j\rangle^{\frac{q}{1-\alpha}(s+\frac{\alpha}{p})} \big\|\Box^{\alpha}_{k,j} f\big\|^q_{L^p(\mathbb{R}^n)} = \|f\|^q_{M^{s+\alpha/p,\alpha}_{p,q}}.$$

Case 1B. $\alpha(n-1) > qs$ and $qs + (1-\alpha)(n-1) > 0$. If $a \geqslant 1$, from (5.10) and $\alpha(n-1) > qs$ we have

$$\Delta_{ja} \lesssim \sum_{k\in\mathscr{K}^n_{j,r_{ja}}} \langle j\rangle^{\frac{\alpha sq}{1-\alpha}+\frac{q\alpha}{p(1-\alpha)}+\alpha(n-1)} \big\|\Box^{\alpha}_{k,j} f\big\|^q_{L^p(\mathbb{R}^n)}. \tag{5.12}$$

Since $qs + (1-\alpha)(n-1) > 0$, similar to (5.9), we see that (5.12) also holds for the case $a = 0$. It follows that

$$\big\|f(\cdot,0)\big\|_{M^{s,\alpha}_{p,q}(\mathbb{R}^{n-1})} \lesssim \|f\|_{M^{\alpha s+(1-\alpha)\frac{\alpha(n-1)}{q}+\alpha/p,\alpha}_{p,q}(\mathbb{R}^n)}.$$

Case 1C. $qs = -(n-1)(1-\alpha)$. Using the first estimate as in (5.9), we have for $a = 0$,

$$\Delta_{ja} \lesssim \sum_{k\in\mathscr{K}^n_{j,r_{ja}}} \langle j\rangle^{\frac{q\alpha}{p(1-\alpha)}} \ln\langle j\rangle \big\|\Box^{\alpha}_{k,j} f\big\|^q_{L^p(\mathbb{R}^n)}. \tag{5.13}$$

For $a \geqslant 1$,

$$\Delta_{ja} \lesssim \sum_{k\in\mathscr{K}^n_{j,r_{ja}}} \langle j\rangle^{\frac{\alpha sq}{1-\alpha}+\frac{q\alpha}{p(1-\alpha)}+\alpha(n-1)} \big\|\Box^{\alpha}_{k,j} f\big\|^q_{L^p(\mathbb{R}^n)}. \tag{5.14}$$

This implies that

$$\big\|f(\cdot,0)\big\|_{M^{s,\alpha}_{p,q}(\mathbb{R}^{n-1})} \lesssim \|f\|_{M^{\alpha/p+\varepsilon,\alpha}_{p,q}(\mathbb{R}^n)}.$$

Case 1D. $qs < -(n-1)(1-\alpha)$. It is easy to see that $\ln\langle j\rangle$ can be removed in (5.13). So, we have the result, as desired.

Case 2. $1 \leqslant p, q < \infty$. Using Minkowski's inequality, we have

$$\big\|f(\cdot,0)\big\|_{M^{s,\alpha}_{p,q}(\mathbb{R}^{n-1})} \lesssim \Bigg(\sum_{l\in\mathbb{Z}_+} \sum_{m\in\mathscr{K}^{n-1}_l} \langle l\rangle^{\frac{sq}{1-\alpha}} \bigg(\sum_{j\in\mathbb{Z}_+} \sum_{k\in\mathscr{K}^n_j} \big\|\big(\mathscr{F}^{-1}_{\bar\xi}\psi_{m,l}\big) * \big(\mathscr{F}^{-1}\psi_{k,j}\mathscr{F}f\big)(\cdot,0)\big\|_{L^p(\mathbb{R}^{n-1})}\bigg)^q\Bigg)^{1/q}$$
$$\lesssim \sum_{j\in\mathbb{Z}_+} \sum_{a=0}^{N_j} \sum_{k\in\mathscr{K}^n_{j,r_{ja}}} \bigg(\sum_{l\in\mathbb{Z}_+} \sum_{m\in\mathscr{K}^{n-1}_l} \langle l\rangle^{\frac{sq}{1-\alpha}} \big\|\big(\mathscr{F}^{-1}_{\bar\xi}\psi_{m,l}\big) * \big(\mathscr{F}^{-1}\psi_{k,j}\mathscr{F}f\big)(\cdot,0)\big\|^q_{L^p(\mathbb{R}^{n-1})}\bigg)^{1/q}. \tag{5.15}$$

Using the same way as in Case 1, we can get that for any $k \in \mathscr{K}^n_{j,r_{ja}}$,

$$\sum_{l\in\mathbb{Z}_+} \sum_{m\in\mathscr{K}^{n-1}_l} \langle l\rangle^{\frac{sq}{1-\alpha}} \big\|\big(\mathscr{F}^{-1}_{\bar\xi}\psi_{m,l}\big) * \big(\mathscr{F}^{-1}\psi_{k,j}\mathscr{F}f\big)(\cdot,0)\big\|^q_{L^p(\mathbb{R}^{n-1})}$$
$$\lesssim \sum_{a^{1-\alpha}\langle j\rangle^{\alpha}\lesssim l\lesssim(1+a)^{1-\alpha}\langle j\rangle^{\alpha}} \langle l\rangle^{\frac{sq}{1-\alpha}} \min\bigg(\langle l\rangle^{n-2}, \frac{\langle j\rangle^{\frac{\alpha(n-2)}{1-\alpha}}}{\langle l\rangle^{\frac{\alpha(n-2)}{1-\alpha}}}\bigg) \langle j\rangle^{\frac{q\alpha}{p(1-\alpha)}} \big\|\Box^{\alpha}_{k,j} f\big\|^q_{L^p(\mathbb{R}^n)}. \tag{5.16}$$

Repeating the calculation procedure as in Case 1, we have the result, as desired.

Case 3. $0 < q \leqslant p < 1$. We have

$$\left\| \mathscr{F}_{\bar{\xi}}^{-1} \psi_{m,l} \mathscr{F}_{\bar{x}} f(\cdot,0) \right\|_{L^p(\mathbb{R}^{n-1})}^p \leqslant \sum_{j \geqslant l-C,\, k \in \mathscr{K}_j^n} \left\| \left(\mathscr{F}_{\bar{\xi}}^{-1} \psi_{m,l} \mathscr{F}_{\bar{x}} \mathscr{F}^{-1} \psi_{k,j} \mathscr{F} f \right)(\cdot,0) \right\|_{L^p(\mathbb{R}^{n-1})}^p. \tag{5.17}$$

It follows that

$$\begin{aligned} \left\| f(\cdot,0) \right\|_{M_{p,q}^{s,\alpha}(\mathbb{R}^{n-1})} &\lesssim \left(\sum_{l \in \mathbb{Z}_+} \sum_{m \in \mathscr{K}_l^{n-1}} \langle l \rangle^{\frac{sq}{1-\alpha}} \left(\sum_{j \in \mathbb{Z}_+} \sum_{k \in \mathscr{K}_j^n} \left\| \left(\mathscr{F}_{\bar{\xi}}^{-1} \psi_{m,l} \right) * \left(\mathscr{F}^{-1} \psi_{k,j} \mathscr{F} f \right)(\cdot,0) \right\|_{L^p(\mathbb{R}^{n-1})}^p \right)^{q/p} \right)^{1/q} \\ &\lesssim \left(\sum_{j \in \mathbb{Z}_+} \sum_{a=0}^{N_j} \sum_{k \in \mathscr{K}_{j,r_{ja}}^n} \sum_{l \leqslant j+C} \sum_{m \in \mathscr{K}_l^{n-1}} \langle l \rangle^{\frac{sq}{1-\alpha}} \left\| \left(\mathscr{F}_{\bar{\xi}}^{-1} \psi_{m,l} \right) * \left(\mathscr{F}^{-1} \psi_{k,j} \mathscr{F} f \right)(\cdot,0) \right\|_{L^p(\mathbb{R}^{n-1})}^q \right)^{1/q}. \end{aligned} \tag{5.18}$$

For convenience, we write

$$\Upsilon_{ja} := \sum_{k \in \mathscr{K}_{j,r_{ja}}^n} \sum_{l \leqslant j+C} \sum_{m \in \mathscr{K}_l^{n-1}} \langle l \rangle^{\frac{sq}{1-\alpha}} \left\| \left(\mathscr{F}_{\bar{\xi}}^{-1} \psi_{m,l} \right) * \left(\mathscr{F}^{-1} \psi_{k,j} \mathscr{F} f \right)(\cdot,0) \right\|_{L^p(\mathbb{R}^{n-1})}^q. \tag{5.19}$$

It follows from Lemma 4.2, the property of p-BAPU and (5.7) that

$$\begin{aligned} \Upsilon_{ja} &\lesssim \sum_{k \in \mathscr{K}_{j,r_{ja}}^n} \sum_{a^{1-\alpha} \langle j \rangle^\alpha \lesssim l \lesssim (1+a)^{1-\alpha} \langle j \rangle^\alpha} \sum_{m \in \mathscr{K}_l^{n-1}} \langle l \rangle^{\frac{sq}{1-\alpha}} \\ &\quad \times \langle j \rangle^{q \frac{\alpha(n-1)}{1-\alpha} (\frac{1}{p}-1)} \left\| \left(\mathscr{F}_{\bar{\xi}}^{-1} \psi_{m,l} \right) \right\|_{L^p(\mathbb{R}^{n-1})}^q \left\| \left(\mathscr{F}^{-1} \psi_{k,j} \mathscr{F} f \right)(\cdot,0) \right\|_{L^p(\mathbb{R}^{n-1})}^q \\ &\lesssim \sum_{k \in \mathscr{K}_{j,r_{ja}}^n} \sum_{a^{1-\alpha} \langle j \rangle^\alpha \lesssim l \lesssim (1+a)^{1-\alpha} \langle j \rangle^\alpha} \langle l \rangle^{\frac{sq}{1-\alpha}} \langle j \rangle^{q \frac{\alpha(n-1)}{1-\alpha} (\frac{1}{p}-1)} \langle l \rangle^{-q \frac{\alpha(n-1)}{1-\alpha} (\frac{1}{p}-1)} \\ &\quad \times \min\left(\langle l \rangle^{n-2}, \langle j \rangle^{\frac{\alpha(n-2)}{1-\alpha}} \langle l \rangle^{-\frac{\alpha(n-2)}{1-\alpha}} \right) \langle j \rangle^{\frac{q\alpha}{p(1-\alpha)}} \left\| \Box_{k,j}^\alpha f \right\|_{L^p(\mathbb{R}^n)}^q. \end{aligned} \tag{5.20}$$

If $a \geqslant 1$, then we have

$$\Upsilon_{ja} \lesssim \sum_{k \in \mathscr{K}_{j,r_{ja}}^n} a^{sq-\alpha(n-1)-q\alpha(n-1)(\frac{1}{p}-1)} \langle j \rangle^{\frac{\alpha qs}{1-\alpha} + q\alpha(n-1)(\frac{1}{p}-1) + \alpha(n-1) + \frac{q\alpha}{p(1-\alpha)}} \left\| \Box_{k,j}^\alpha f \right\|_{L^p(\mathbb{R}^n)}^q. \tag{5.21}$$

If $a = 0$,

$$\Upsilon_{ja} \lesssim \sum_{k \in \mathscr{K}_{j,r_{ja}}^n} \sum_{0 \leqslant l \lesssim \langle j \rangle^\alpha} \langle l \rangle^{\frac{sq}{1-\alpha} - q\frac{\alpha(n-1)}{1-\alpha}(\frac{1}{p}-1) + (n-2)} \langle j \rangle^{q\frac{\alpha(n-1)}{1-\alpha}(\frac{1}{p}-1) + \frac{q\alpha}{p(1-\alpha)}} \left\| \Box_{k,j}^\alpha f \right\|_{L^p(\mathbb{R}^n)}^q. \tag{5.22}$$

Now we divide our discussion into the following four subcases.

Case 3A. $qs - \alpha(n-1) - q\alpha(n-1)(1/p-1) \geqslant 0$. If $a \geqslant 1$, we see that the upper bound in (5.21) will be attained at $a \sim \langle j \rangle$. If $a = 0$, the summation on l in (5.22) can be easily controlled. Anyway, we have

$$\Upsilon_{ja} \lesssim \sum_{k \in \mathscr{K}_{j,r_{ja}}^n} \langle j \rangle^{\frac{\alpha qs}{1-\alpha} + \frac{q\alpha}{p(1-\alpha)}} \left\| \Box_{k,j}^\alpha f \right\|_{L^p(\mathbb{R}^n)}^q. \tag{5.23}$$

Combining (5.18) and (5.23), we immediately have the result, as desired.

Case 3B. $q\alpha s_p - (1-\alpha)(n-1) < qs < q\alpha s_p + \alpha(n-1)$. In this case, using (5.21) and (5.22), we can repeat the procedure as in Case 1B to get the result and we omit the details of the proof.

Case 3C. $q\alpha s_p - (1-\alpha)(n-1) = qs$. This case is similar to Case 1C.

Case 3D. $q\alpha s_p - (1-\alpha)(n-1) > qs$. We can deal with this case by following the same way as in Case 1D.

Case 4. $0 < p < 1$, $q > p$. By Minkowski's inequality, we have

$$\begin{aligned}\|f(\cdot,0)\|_{M^{s,\alpha}_{p,q}(\mathbb{R}^{n-1})} &\lesssim \left(\sum_{l\in\mathbb{Z}_+}\sum_{m\in\mathscr{K}_l^{n-1}} \langle l\rangle^{\frac{sq}{1-\alpha}}\left(\sum_{j\in\mathbb{Z}_+}\sum_{k\in\mathscr{K}_j^n}\left\|\left(\mathscr{F}_{\bar{\xi}}^{-1}\psi_{m,l}\right)*\left(\mathscr{F}^{-1}\psi_{k,j}\mathscr{F}f\right)(\cdot,0)\right\|^p_{L^p(\mathbb{R}^{n-1})}\right)^{q/p}\right)^{1/q}\\ &\lesssim \left(\sum_{j\in\mathbb{Z}_+}\sum_{k\in\mathscr{K}_j^n}\left(\sum_{l\in\mathbb{Z}_+}\sum_{m\in\mathscr{K}_l^{n-1}} \langle l\rangle^{\frac{sq}{1-\alpha}}\left\|\left(\mathscr{F}_{\bar{\xi}}^{-1}\psi_{m,l}\right)*\left(\mathscr{F}^{-1}\psi_{k,j}\mathscr{F}f\right)(\cdot,0)\right\|^q_{L^p(\mathbb{R}^{n-1})}\right)^{p/q}\right)^{1/p}.\end{aligned} \tag{5.24}$$

Then we can repeat the procedures as in the proof of Theorem 3.1 and the above techniques in Case 3 to have the result, as desired. The details of the proof are omitted. □

6. Proof of Theorem 3.5

Now we prove Theorem 3.5. Now we define the maximum function $M_k^* f$ as follows:

$$M_k^* f = \sup_{y\in\mathbb{Z}^n}\frac{|\Delta_k f(x-y)|}{1+|2^k y|^{n/r}}. \tag{6.1}$$

Taking $y_1=\cdots=y_{n-1}=0$, $y_n=x_n$ in (6.1), we have for $2^{-k-1}\leqslant |x_n|\leqslant 2^{-k}$,

$$\left|(\Delta_k f)(\bar{x},0)\right| \lesssim \left|M_k^* f(x)\right|, \quad \bar{x}=(x_1,\dots,x_{n-1}).$$

Hence

$$\left\|(\Delta_k f)(\cdot,0)\right\|_{L^p(\mathbb{R}^{n-1})} \lesssim \left\|M_k^* f(\cdot,x_n)\right\|_{L^p(\mathbb{R}^{n-1})}. \tag{6.2}$$

Integrating (6.2), one has that

$$\left\|(\Delta_k f)(\cdot,0)\right\|^p_{L^p(\mathbb{R}^{n-1})} \lesssim 2^k\int_{\mathbb{R}}\left\|M_{k,t}^* f(\cdot,x_n)\right\|^p_{L^p(\mathbb{R}^{n-1})}\,dx_n.$$

Hence

$$\left\|(\Delta_k f)(\cdot,0)\right\|_{L^p(\mathbb{R}^{n-1})} \lesssim 2^{k/p}\left\|M_k^* f\right\|_{L^p(\mathbb{R}^n)}. \tag{6.3}$$

Write $\varphi_k'(\bar{x})$ as the BAPU functions in $\mathbb{R}^{n-1}$. Then for fixed k, we have

$$\left(\mathscr{F}_{\bar{\xi}}^{-1}\varphi_k'\mathscr{F}_{\bar{x}}f\right)(\bar{x},0)=\sum_{l=k-1}^{\infty}\left(\mathscr{F}_{\bar{\xi}}^{-1}\varphi_k'\right)*\left(\mathscr{F}^{-1}\varphi_l\mathscr{F}f\right)(\bar{x},0).$$

Case 1. $1\leqslant p<\infty$. Using Young's inequality, (4.2) and (6.3), we obtain

$$\begin{aligned}\left\|\left(\mathscr{F}_{\bar{\xi}}^{-1}\varphi_k'\mathscr{F}_{\bar{x}}\right)f(\cdot,0)\right\|_{L^p(\mathbb{R}^{n-1})} &\lesssim \sum_{l=k-1}^{\infty}\left\|\mathscr{F}_{\bar{\xi}}^{-1}\varphi_k'\right\|_{L^1(\mathbb{R}^{n-1})}\left\|\mathscr{F}^{-1}\varphi_l\mathscr{F}f\right\|_{L^p(\mathbb{R}^{n-1})}\\ &\lesssim \sum_{l=k-1}^{\infty}\left\|M_l^* f\right\|_{L^p(\mathbb{R}^n)}\\ &\lesssim \sum_{l=k-1}^{\infty}2^{l/p}\|\Delta_l f\|_{L^p(\mathbb{R}^n)}.\end{aligned}$$

Hence,

$$\left\|f(\cdot,0)\right\|_{B^s_{p,q}(\mathbb{R}^{n-1})} \lesssim \left(\sum_{k=0}^{\infty}2^{skq}\left(\sum_{l=k-1}^{\infty}2^{l/p}\|\Delta_l f\|_{L^p(\mathbb{R}^n)}\right)^q\right)^{1/q}.$$

If $0<q\leqslant 1$, then

$$\left\|f(\cdot,0)\right\|_{B^s_{p,q}(\mathbb{R}^{n-1})} \lesssim \left(\sum_{l=-1}^{\infty}\sum_{k=0}^{l+1}2^{skq}2^{lq/p}\|\Delta_l f\|^q_{L^p(\mathbb{R}^n)}\right)^{1/q}.$$

If $s=0$, then

$$\|f(\cdot,0)\|_{B^s_{p,q}(\mathbb{R}^{n-1})} \lesssim \left(\sum_{l=-1}^{\infty} l2^{lq/p}\|\Delta_l f\|^q_{L^p(\mathbb{R}^n)}\right)^{1/q} \lesssim \|f\|_{\tilde{B}^{1/p}_{p,q}(\mathbb{R}^n)}.$$

In the case $s<0$,

$$\|f(\cdot,0)\|_{B^s_{p,q}(\mathbb{R}^{n-1})} \lesssim \left(\sum_{l=-1}^{\infty} 2^{lq/p}\|\Delta_l f\|^q_{L^p(\mathbb{R}^n)}\right)^{1/q} \lesssim \|f\|_{B^{1/p}_{p,q}(\mathbb{R}^n)}.$$

If $1\leqslant q<\infty$, using Minkowski's inequality,

$$\|f(\cdot,0)\|_{B^s_{p,q}(\mathbb{R}^{n-1})} \lesssim \sum_{l=-1}^{\infty}\left(\sum_{k=0}^{l+1} 2^{skq}2^{lq/p}\|\Delta_l f\|^q_{L^p(\mathbb{R}^n)}\right)^{1/q},$$

then

$$\|f(\cdot,0)\|_{B^s_{p,q}(\mathbb{R}^{n-1})} \lesssim \begin{cases} \|f\|_{\tilde{B}^{1/p}_{p,1}(\mathbb{R}^n)} & s=0,\\ \|f\|_{B^{1/p}_{p,1}(\mathbb{R}^n)} & s<0.\end{cases}$$

Case 2. $0<p<1$.

$$\begin{aligned}\left\|\left(\mathscr{F}^{-1}_{\bar{\xi}}\varphi'_k\mathscr{F}_{\bar{x}}\right)f(\cdot,0)\right\|^p_{L^p(\mathbb{R}^{n-1})} &\lesssim \sum_{l=k-1}^{\infty} 2^{l(n-1)(1/p-1)p}\left\|\mathscr{F}^{-1}_{\bar{\xi}}\varphi'_k\right\|^p_{L^p(\mathbb{R}^{n-1})}\left\|\mathscr{F}^{-1}\varphi_l\mathscr{F}f\right\|^p_{L^p(\mathbb{R}^{n-1})}\\ &\lesssim \sum_{l=k-1}^{\infty} 2^{l(n-1)(1-p)}2^{k(n-1)(p-1)}2^l\left\|M^*_l f\right\|^p_{L^p(\mathbb{R}^n)}\\ &\lesssim \sum_{l=k-1}^{\infty} 2^{l(n-1)(1-p)}2^{k(n-1)(p-1)}2^l\|\Delta_l f\|^p_{L^p(\mathbb{R}^n)}.\end{aligned}$$

It follows that

$$\|f(\cdot,0)\|_{B^s_{p,q}(\mathbb{R}^{n-1})} \lesssim \left(\sum_{k=0}^{\infty}2^{skq}\left(\sum_{l=k-1}^{\infty} 2^{l(n-1)(1-p)}2^{k(n-1)(p-1)}2^l\|\Delta_l f\|^p_{L^p(\mathbb{R}^n)}\right)^{q/p}\right)^{1/q}.$$

If $q\leqslant p$, one has that

$$\begin{aligned}\|f(\cdot,0)\|^q_{B^s_{p,q}(\mathbb{R}^{n-1})} &\lesssim \sum_{k=0}^{\infty}\sum_{l=k-1}^{\infty} 2^{l(n-1)(1/p-1)q}2^{k(s+(n-1)(1-1/p))q}2^{lq/p}\|\Delta_l f\|^q_{L^p(\mathbb{R}^n)}\\ &\lesssim \sum_{l=-1}^{\infty}\sum_{k=0}^{l+1} 2^{l(n-1)(1/p-1)q}2^{k(s+(n-1)(1-1/p))q}2^{lq/p}\|\Delta_l f\|^q_{L^p(\mathbb{R}^n)}.\end{aligned}$$

Recall that we write $s_p=(n-1)(1/(p\wedge 1)-1)$, then

$$\|f(\cdot,0)\|_{B^s_{p,q}(\mathbb{R}^{n-1})} \lesssim \begin{cases} \|f\|_{\tilde{B}^{s_p+1/p}_{p,q}(\mathbb{R}^n)}, & s=s_p,\\ \|f\|_{B^{s_p+1/p}_{p,q}(\mathbb{R}^n)}, & s<s_p.\end{cases}$$

If $q\geqslant p$, using Minkowski's inequality, we have

$$\begin{aligned}\|f(\cdot,0)\|_{B^s_{p,q}(\mathbb{R}^{n-1})} &\lesssim \left[\sum_{l=-1}^{\infty}\left(\sum_{k=0}^{\infty}\left(\chi(k\leqslant l)2^{l(n-1)(1-p)}2^{k(s+(n-1)(p-1))}2^l\|\Delta_l f\|^p_{L^p(\mathbb{R}^n)}\right)^{q/p}\right)^{p/q}\right]^{1/p}\\ &\lesssim \left[\sum_{l=-1}^{\infty}\sum_{k=0}^{l+1} 2^{l(n-1)(1/p-1)p}2^{k(s+(n-1)(1-1/p))p}2^l\|\Delta_l f\|^q_{L^p(\mathbb{R}^n)}\right]^{1/p}.\end{aligned}$$

Therefore,

$$\|f(\cdot,0)\|_{B^s_{p,q}(\mathbb{R}^{n-1})} \lesssim \begin{cases} \|f\|_{\tilde{B}^{s_p+1/p}_{p,p}(\mathbb{R}^n)}, & s=s_p, \\ \|f\|_{B^{s_p+1/p}_{p,p}(\mathbb{R}^n)}, & s<s_p. \end{cases}$$

In the case $1<p<\infty$, we define the maximum function $M^*_{k,t}f$ as follows:

$$M^*_{k,t}f = \sup_{y\in\mathbb{Z}^n} \frac{|\Delta_{k,t}f(x-y)|}{1+|2^k y|^{n/r}}. \tag{6.4}$$

Taking $y_1=\cdots=y_{n-1}=0$, $y_n=x_n$ in (6.4), we have for $2^{-k-1}\leqslant |x_n|\leqslant 2^{-k}$,

$$\left|(\Delta_{k,t}f)(\bar{x},0)\right| \lesssim \left|M^*_{k,t}f(x)\right|, \quad \bar{x}=(x_1,\dots,x_{n-1}).$$

Hence

$$\left\|(\Delta_{k,t}f)(\cdot,0)\right\|_{L^p(\mathbb{R}^{n-1})} \lesssim \left\|M^*_{k,t}f(\cdot,x_n)\right\|_{L^p(\mathbb{R}^{n-1})}. \tag{6.5}$$

Integrating (6.5), one has that

$$\left\|(\Delta_{k,t}f)(\cdot,0)\right\|^p_{L^p(\mathbb{R}^{n-1})} \lesssim 2^k \int_{\mathbb{R}} \left\|M^*_{k,t}f(\cdot,x_n)\right\|^p_{L^p(\mathbb{R}^{n-1})}\,dx_n.$$

Hence

$$\left\|(\Delta_{k,t}f)(\cdot,0)\right\|_{L^p(\mathbb{R}^{n-1})} \lesssim 2^{k/p}\left\|M^*_{k,t}f\right\|_{L^p(\mathbb{R}^n)}. \tag{6.6}$$

Let $\chi'_{k,t}(\bar{x})$ as the characteristic functions in $\mathbb{R}^{n-1}$. Then for fixed k and t, we have

$$\left(\mathscr{F}^{-1}_{\bar{\xi}}\chi'_{k,t}\mathscr{F}_{\bar{x}}f\right)(\bar{x},0) = \sum_{l=k}^{\infty}\left(\mathscr{F}^{-1}_{\bar{\xi}}\chi'_{k,t}\right) * \left(\mathscr{F}^{-1}\chi_{l,t}\mathscr{F}f\right)(\bar{x},0).$$

For $l=k$, $\operatorname{supp}\chi_{l,t}\cap\operatorname{supp}\chi'_{k,t}\neq\varnothing$ implies that $t=2^n+1,\dots,T$, and for $l\geqslant 1+k$, $\operatorname{supp}\chi_{l,t}\cap\operatorname{supp}\chi'_{k,t}\neq\varnothing$ implies that $t=1,\dots,2^n$. Using Young's inequality, (4.2) and (6.6), we obtain

$$\begin{aligned} \left\|\mathscr{F}^{-1}_{\bar{\xi}}\chi'_{k,t}\mathscr{F}_{\bar{x}}f(\bar{x},0)\right\|_{L^p(\mathbb{R}^{n-1})} &\lesssim \sum_{l=k+1}^{\infty}\left\|M^*_{l,t}f\right\|_{L^p(\mathbb{R}^n)}\chi_{\{t=1,\dots,2^n\}} + \left\|M^*_{k,t}f\right\|_{L^p(\mathbb{R}^n)}\chi_{\{t=2^n+1,\dots,T\}} \\ &\lesssim \sum_{l=k+1}^{\infty} 2^{l/p}\|\Delta_{l,t}f\|_{L^p(\mathbb{R}^n)}\chi_{\{t=1,\dots,2^n\}} + 2^{l/p}\|\Delta_{k,t}f\|_{L^p(\mathbb{R}^n)}\chi_{\{t=2^n+1,\dots,T\}}. \end{aligned}$$

Hence,

$$\left\|f(\cdot,0)\right\|_{B^s_{p,q}(\mathbb{R}^{n-1})} \lesssim \left(\sum_{k=0}^{\infty}2^{skq}\left(\sum_{l=k+1}^{\infty}2^{l/p}\sum_{t=1}^{2^n}\|\Delta_{l,t}f\|_{L^p(\mathbb{R}^n)} + \sum_{t=2^n+1}^{T}\|\Delta_{k,t}f\|_{L^p(\mathbb{R}^n)}\right)^q\right)^{1/q}.$$

If $0<q\leqslant 1$, then

$$\left\|f(\cdot,0)\right\|_{B^s_{p,q}(\mathbb{R}^{n-1})} \lesssim \left(\sum_{l=0}^{\infty}\sum_{k=0}^{l-1}\sum_{t=1}^{2^n}2^{skq}2^{lq/p}\|\Delta_{l,t}f\|^q_{L^p(\mathbb{R}^n)} + \sum_{l=0}^{\infty}\sum_{t=2^n+1}^{T}2^{slq}2^{lq/p}\|\Delta_{l,t}f\|^q_{L^p(\mathbb{R}^n)}\right)^{1/q}.$$

If $s=0$, then

$$\begin{aligned} \left\|f(\cdot,0)\right\|_{B^s_{p,q}(\mathbb{R}^{n-1})} &\lesssim \left(\sum_{l=0}^{\infty}\sum_{t=1}^{2^n} l2^{lq/p}\|\Delta_{l,t}f\|^q_{L^p(\mathbb{R}^n)} + \sum_{l=0}^{\infty}\sum_{t=2^n+1}^{T}2^{lq/p}\|\Delta_{l,t}f\|^q_{L^p(\mathbb{R}^n)}\right)^{1/q} \\ &\lesssim \|f\|_{\tilde{B}^{1/p,1/p}_{p,q}}. \end{aligned}$$

In the case $s<0$,

$$\begin{aligned} \left\|f(\cdot,0)\right\|_{B^s_{p,q}(\mathbb{R}^{n-1})} &\lesssim \left(\sum_{l=0}^{\infty}\sum_{t=1}^{2^n}2^{lq/p}\|\Delta_{l,t}f\|^q_{L^p(\mathbb{R}^n)} + \sum_{l=0}^{\infty}\sum_{t=2^n+1}^{T}2^{slq}2^{lq/p}\|\Delta_{l,t}f\|^q_{L^p(\mathbb{R}^n)}\right)^{1/q} \\ &\lesssim \|f\|_{B^{1/p,s+1/p}_{p,q}}. \end{aligned}$$

If $1 \leqslant q < \infty$, using Minkowski's inequality,

$$\|f(\cdot,0)\|_{B^s_{p,q}(\mathbb{R}^{n-1})} \lesssim \sum_{l=0}^{\infty}\left(\sum_{k=0}^{l-1}\sum_{t=1}^{2^n} 2^{skq}2^{lq/p}\|\Delta_{l,t}f\|^q_{L^p(\mathbb{R}^n)} + \sum_{t=2^n+1}^{T} 2^{slq}2^{lq/p}\|\Delta_{l,t}f\|^q_{L^p(\mathbb{R}^n)}\right)^{1/q},$$

then

$$\|f(\cdot,0)\|_{B^s_{p,q}(\mathbb{R}^{n-1})} \lesssim \begin{cases} \|f\|_{\tilde{B}^{1/p,1/p}_{p,1}(\mathbb{R}^n)} & s=0, \\ \|f\|_{B^{1/p,s+1/p}_{p,1}(\mathbb{R}^n)} & s<0. \end{cases} \qquad \square$$

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